Some noteworthy spin plethysms

1982 J. Phys. A: Math. Gen. 15 1137

(http://iopscience.iop.org/0305-4470/15/4/017)

J. Phys. A: Math. Gen. **15** (1982) 1137–1141. Printed in Great Britain

Some noteworthy spin plethysms

R C King and B G Wybourne†

Mathematics Department, The University, Southampton, SO9 5NH, UK

Received 25 September 1981

Abstract. The spin plethysms $\lambda_G \otimes \Delta$ that arise in the reduction of Δ under $SO(N) \to G$ when $[1] \to \lambda_G$ are considered. It is shown that, for the simple Lie algebras of rank k, if $\lambda_G = \varphi_G$, the adjoint representation of G, then $\varphi_G \otimes \Delta = 2^{[k/2]}\delta_G$ where δ_G is the representation of G whose highest weight is half the sum of the positive roots. Certain results for other representations are described. A remarkable series of $\mathcal{S}$-functions is introduced leading to a new dimensional equality between certain representations of O($2k$) and Sp($2k$).

The adjoint representation φ_G of each simple Lie group G is orthogonal and unimodular (Malćev 1962). It follows that G may be embedded in SO(N) where N is the dimension of φ_G. This embedding, signified by

$$SO(N) \supset G \qquad [1] \to \varphi_G, \tag{1}$$

is such that the branching rule for any representation λ of SO(N) takes the form

$$SO(N) \supset G \qquad \lambda \to \varphi_G \otimes \lambda \tag{2}$$

where $\varphi_G \otimes \lambda$ denotes a plethysm (Littlewood 1950) whose total dimension is equal to that of λ. In particular, the branching of the spin representation Δ of SO(N), of dimension $2^{[N/2]}$, is given by

$$SO(N) \supset G \qquad \Delta \to \varphi_G \otimes \Delta. \tag{3}$$

The evaluation of the spin plethysm $\varphi_G \otimes \Delta$ may be accomplished by considering the mapping from the weights of the representation [1] of SO(N) to the weights of the representation φ_G of G. These latter weights are simply the roots $\pm \boldsymbol{r}(\alpha)$ of the corresponding Lie algebra g of dimension N, together with k null vectors $\mathbf{0}$, where k is the rank of g. The number of positive roots $\boldsymbol{r}(\alpha)$ of g is $(N-k)/2$.

The order-preserving map corresponding to (1) then takes the form

$$\boldsymbol{m}(i) \to \begin{cases} \boldsymbol{r}(\alpha) & \text{for } i = \alpha = 1, 2, \ldots, (N-k)/2 \\ \mathbf{0} & \text{for } i = (N-k)/2+1, (N-k)/2+2, \ldots, (N+k)/2-1 \\ -\boldsymbol{r}(\alpha) & \text{for } i = N-\alpha+1 = (N+k)/2, (N+k)/2+1, \ldots, N-1, N, \end{cases} \tag{4}$$

where, in the $[N/2]$-dimensional weight space of SO(N),

$$\boldsymbol{m}(i) = \begin{cases} \boldsymbol{e}_i & \text{for } i = 1, 2, \ldots, [N/2] \\ \mathbf{0} & \text{for } i = [N/2]+1 \text{ if } N \text{ is odd} \\ -\boldsymbol{e}_{N-i+1} & \text{for } i = N-[N/2]+1, \ldots, N-1, N \end{cases} \tag{5}$$

with $\boldsymbol{e}_i = (00\ldots1\ldots0)$ where the ith component is 1 and all others vanish.

† Permanent address: Physics Department, University of Canterbury, Christchurch, New Zealand.

The weights of the spin representations Δ of SO(N) are the $2^{[N/2]}$ vectors

$$w = \sum_{i=1}^{[N/2]} \eta_i m(i) \tag{6}$$

with $\eta_i = \pm\frac{1}{2}$.

The mapping of the highest weight then takes the form

$$(\tfrac{1}{2}, \tfrac{1}{2}, \ldots, \tfrac{1}{2}) \rightarrow \tfrac{1}{2}\sum_{\alpha} r(\alpha) = \delta \tag{7}$$

where δ is half the sum of the positive roots of g. This weight is the highest weight of an irreducible representation δ_G of G which is thus necessarily a constituent of the restriction to G of Δ.

More generally this weight vector δ is produced under the mapping (4) from the weights (6) of Δ in precisely $2^{[k/2]}$ ways, since the last $[k/2]$ coefficients η_i may be changed from $+\frac{1}{2}$ to $-\frac{1}{2}$ without altering the image of w under (4). It follows that

$$\varphi_G \otimes \Delta \supset 2^{[k/2]}\delta_G.$$

The dimension of the irreducible representation λ_G is given by Weyl's character formula

$$d(\lambda_G) = \prod_{r(\alpha)>0} r(\alpha)\cdot(\lambda+\delta) \Big/ \prod_{r(\alpha)>0} r(\alpha)\cdot\delta \tag{8}$$

where λ is the highest weight of λ_G. Hence

$$d(\delta_G) = 2^{(N-k)/2}. \tag{9}$$

However

$$d(\Delta) = 2^{[N/2]}. \tag{10}$$

Since $N \equiv k \pmod 2$ for each simple Lie algebra, a dimensional check is sufficient to confirm the following general result:

$$\mathrm{SO}(N) \supset G \qquad [1] \rightarrow \varphi_G \qquad \Delta \rightarrow \varphi_G \otimes \Delta = 2^{[k/2]}\delta_G. \tag{11}$$

Of course, if N is even, the spin representation Δ has two irreducible constituents and the branching rule is then

$$\Delta_{\pm} \rightarrow \varphi_G \otimes \Delta_{\pm} = 2^{[k/2]-1}\delta_G. \tag{12}$$

The representation δ_G of G, whose highest weight is half the sum of the positive roots, is specified in the Dynkin notation by attaching a 1 to each circle of the Dynkin diagram. This follows from the fact that δ is the sum of the highest weights of the k elementary representations of G (Dynkin 1957, p 356).

The representations φ_G and δ_G are specified, along with their dimensions, in table 1 in a notation (Wybourne and Bowick 1977, King and Al-Qubanchi 1981) developed more recently.

It is of interest to determine to what extent the result obtained for the spin plethysm $\varphi_G \otimes \Delta$ depends upon the fact that φ_G is the adjoint representation of G.

If ψ_G is any orthogonal, unimodular representation of G of dimension M, then G may be embedded in SO(M) with the embedding defined by

$$\mathrm{SO}(M) \supset G \qquad [1] \rightarrow \psi_G.$$

Table 1.

g	G	φ_G	$d(\varphi_G)=N$	δ_G	$d(\delta_G)$
A_k	$SU(k+1)$	$\{21^{k-1}\}$	$k(k+2)$	$\{k, k-1, \ldots, 1\}$	$2^{k(k+1)/2}$
B_k	$SO(2k+1)$	$[1^2]$	$k(2k+1)$	$[\Delta; k-1, k-2, \ldots, 1, 0]$	2^{k^2}
C_k	$Sp(2k)$	$\langle 2\rangle$	$k(2k+1)$	$\langle k, k-1, \ldots, 1\rangle$	2^{k^2}
D_k	$SO(2k)$	$[1^2]$	$k(2k-1)$	$[k-1, k-2, \ldots, 1, 0]$	$2^{k(k-1)}$
G_2	G_2	(21)	14	(31)	2^6
F_4	F_4	(1^2)	52	$(\Delta; 521)$	2^{24}
E_6	E_6	$(2;0)$	78	$(11; 54321)$	2^{36}
E_7	E_7	(21^6)	133	$(17,654321)$	2^{63}
E_8	E_8	(1^2)	248	$(29,7654321)$	2^{120}

The branching rule for the spin representation Δ of $SO(M)$ is given by

$$SO(M)\supset G \qquad \Delta\to\psi_G\otimes\Delta,$$

and the spin plethysm $\psi_G\otimes\Delta$ may be evaluated as before by considering the images of the weights of Δ under the mappings of the weights of [1] to those of ψ_G. The analogue of (7) is then

$$(\tfrac{1}{2}\tfrac{1}{2}\ldots\tfrac{1}{2})\to\tfrac{1}{2}\sum_{\boldsymbol{m}>0} M_{\boldsymbol{m}}^{\psi_G}\boldsymbol{m}=\boldsymbol{\delta}^{\psi_G}. \tag{13}$$

Here $M_{\boldsymbol{m}}^{\psi_G}$ is the multiplicity of the weight $\boldsymbol{m}$ in the representation ψ_G, so that $\boldsymbol{\delta}^{\psi_G}$ is half the sum of the positive weights of ψ_G. This weight is the highest weight of an irreducible representation $\delta_G^{\psi_G}$ of G. Making use of (6) it follows that

$$\psi_G\otimes\Delta\supset 2^{[k^{\psi_G}/2]}\delta_G^{\psi_G}$$

where $k^{\psi_G}=M_{\boldsymbol{0}}^{\psi_G}$ is the multiplicity of the null weight in ψ_G.

For arbitrary ψ_G this branching rule is not complete, but there are special cases, $\psi_G=\varphi_G^{w}$, for which

$$\varphi_G^{w}\otimes\Delta=2^{[k^w/2]}\delta_G^{w}. \tag{14}$$

For convenience, in these cases, k^{ψ_G} and $\delta_G^{\psi_G}$ have been denoted by k^w and δ_G^w, respectively. Examples of such cases are provided by $\varphi_G^w=[2]$, $\langle 1^2\rangle$ or $[2]$ of $SO(2k+1)$, $Sp(2k)$ or $O(2k)$, respectively. The corresponding branching rules are

$$SO(M)\supset SO(2k+1) \qquad \Delta\to[2]\otimes\Delta=2^{[k/2]}[\Delta; k, k-1, \ldots, 2, 1] \tag{15a}$$

$$SO(M)\supset Sp(2k) \qquad \Delta\to\langle 1^2\rangle\otimes\Delta=2^{[(k-1)/2]}\langle k-1, k-2, \ldots, 1, 0\rangle \tag{15b}$$

$$SO(M)\supset O(2k)\supset SO(2k) \qquad \Delta\to[2]\otimes\Delta=2^{[(k-1)/2]}[k, k-1, \ldots, 2, 1]$$

$$\to 2^{[(k-1)/2]}([k, k-1, \ldots, 2, 1]_+ + [k, k-1, \ldots, 2, 1]_-). \tag{15c}$$

The validity of the first two results is confirmed by a dimensionality check based on the identity (El-Samra and King 1979)

$$d_{2k+1}[\Delta; \lambda]=2^k d_{2k}\langle\lambda\rangle, \tag{16}$$

together with the known results for $d_{2k}\langle k, k-1, \ldots, 2, 1\rangle$ and $d_{2k}[\Delta; k-1, k-2, \ldots, 1, 0]$ already given in table 1.

In order to deal with the third result (15*c*), it is convenient to introduce a truly remarkable series of $\mathscr{S}$-functions

$$T=\sum_{\tau}\{\tau\} \tag{17}$$

where the sum is taken over all partitions (τ) having Frobenius symbols of the form

$$(\tau)=\begin{pmatrix} a & a-2 & a-4 & \dots \\ a & a-2 & a-4 & \dots \end{pmatrix}=(a+1, a, \dots, 2, 1). \tag{18}$$

The structure of the corresponding Young diagram makes it clear that $\{\tau/m\}=\{\tau/1^m\}$ for all m and hence that

$$\{\tau/M\}=\{\tau/Q\} \qquad \text{and} \qquad \{\tau/L\}=\{\tau/P\} \tag{19}$$

where $LM=PQ=1$.

The following $\mathscr{S}$-function series identities (King *et al* 1981)

$$A=PMC \qquad B=LQD \qquad V=LQ \qquad W=PM \tag{20}$$

then imply that

$$\{\tau/A\}=\{\tau/C\} \qquad \{\tau/B\}=\{\tau/D\} \tag{21}$$

and

$$\{\tau/V\}=\{\tau/W\}=\{\tau\}. \tag{22}$$

The significance of this result is revealed by the relationship between orthogonal and symplectic group characters given by

$$\mathrm{O}(2k)\subset \mathrm{U}(2k)\supset \mathrm{Sp}(2k) \qquad [\lambda]\rightarrow\{\lambda/C\}\rightarrow\langle\lambda/BC\rangle=\langle\lambda/V\rangle. \tag{23}$$

Hence for the special representations labelled by τ

$$d_{2k}[\tau]=d_{2k}\langle\tau\rangle. \tag{24}$$

Thus

$$d_{2k}[k, k-1, \dots, 2, 1]=d_{2k}\langle k, k-1, \dots, 2, 1\rangle=2^{k^2}, \tag{25}$$

confirming the third result. Information concerning φ_G^{w} and δ_G^{w} is given in table 2. Just as in (12), if M is even, the branching rule for $\Delta_\pm$ is that of Δ divided by 2.

As far as further possibilities are concerned, consideration of the defining representations of G_2 and F_4 yields

$$\mathrm{SO}(7)\supset \mathrm{G}_2 \qquad \Delta\rightarrow(1)\otimes\Delta=(1)+(0)$$

$$\mathrm{SO}(26)\supset \mathrm{F}_4 \qquad \Delta_\pm\rightarrow(1)\otimes\Delta_\pm=(\Delta; 2).$$

Table 2.

g	G	φ_G^{w}	$d(\varphi_G^{\mathrm{w}})=M$	δ_G^{w}	$d(\delta_G^{\mathrm{w}})$
B_k	$\mathrm{SO}(2k+1)$	$[2]$	$k(2k+3)$	$[\Delta; k, k-1, \dots, 2, 1]$	$2^{k(k+1)}$
C_k	$\mathrm{Sp}(2k)$	$\langle 1^2\rangle$	$(k-1)(2k+1)$	$\langle k-1, k-2, \dots, 1, 0\rangle$	$2^{k(k-1)}$
D_k	$\mathrm{SO}(2k)$	$[2]$	$(k+1)(2k-1)$	$[k, k-1, \dots, 2, 1]_+ + [k, k-1, \dots, 2, 1]_-$	2^{k^2}

The defining representations of E_6 and E_7 are complex and symplectic, respectively, while that of E_8 is nothing other than the adjoint representation already discussed.

Further simple results seem unlikely. For example, the embedding of G_2 in SO(27) by means of the representation (2) is not maximal,

$$SO(27) \supset SO(7) \supset G_2 \qquad [1] \to [2] \to (2),$$

and the spin plethysm takes a non-trivial form

$$\begin{aligned}\Delta \to 2[\Delta; 321] \to (2) \otimes \Delta \\ &= 2(61) + 2(62) + 2(50) + 4(51) + 2(52) + 2(40) + 4(41) \\ &\quad + 2(42) + 2(30) + 4(31) + 2(20) + 2(21).\end{aligned}$$

The results obtained here were stimulated by the remarkable work of Morris (1961) who derived the particular results appropriate to the cases $G = O(N)$. In addition, the result appropriate to $G = G_2$ is contained in the tables of branching rules given by McKay and Patera (1981) which are not, however, extensive enough for the general results (11) and (14) to be manifest.

Acknowledgment

One of us (BGW) is grateful to the University of Canterbury for the award of an Erskine Fellowship which made this collaboration possible.

References

Dynkin E B 1957 *Am. Math. Soc. Transl. Ser. 2* **6** 245
El-Samra N and King R C 1979 *J. Phys. A: Math. Gen.* **12** 2316
King R C and Al-Qubanchi A H A 1981 *J. Phys. A: Math. Gen.* **14** 15
King R C, Luan Dehuai and Wybourne B G 1981 *J. Phys. A: Math. Gen.* **14** 2509
Littlewood D E 1950 *The Theory of Group Characters* 2nd edn (Oxford: Clarendon)
Malćev A I 1962 *Am. Math. Soc. Transl. Ser. 1* **9** 172
McKay W G and Patera J 1981 *Tables of dimension, indices, and branching rules for representations of simple Lie algebras* (New York: Marcel Dekker)
Morris A O 1961 *Qt. J. Math. (Oxford) (2)* **12** 69
Wybourne B G and Bowick M J 1977 *Austr. J. Phys.* **30** 259